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On the universality of distribution of ranked cluster masses at critical percolation

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Abstract. The distribution of masses of clusters smaller than the infinite cluster is evaluated at the percolation threshold. The clusters are ranked according to their masses and the distribution $P(M/L^D, r)$ of the scaled masses M for any rank r shows a universal behaviour for different lattice sizes L (D is the fractal dimension). For different ranks however, there is a universal distribution function only in the large-rank limit, i.e. $P(M/L^D, r)r^{-y\zeta} \sim g(Mr^y/L^D)$ (y and ζ are defined in the text), where the universal scaling function g is found to be Gaussian in nature.

Percolation is a classic example of systems with quenched disorder [1]. In a discrete lattice, sites or bonds are present with a certain probability and clusters are formed by connecting neighbouring occupied sites. At a critical probability, an 'infinite' cluster appears for the first time which spans the whole lattice.

The average mass or size of the spanning cluster is known to scale as $M \sim L^D$, where L is the lattice size and D the fractal dimension. In two recent papers [2, 3] it was shown that, when the clusters are ranked, the average masses of the ranked clusters also show a similar scaling behaviour. This is true even for the clusters of large rank, which are definitely smaller than the spanning cluster. These clusters have been termed 'effectively spanning' since their masses diverge with the lattice size although they do not really span the lattice. The behaviour of the average scaled mass M/L^D as a function of the rank r was found to be

$$\langle M/L^D \rangle \sim r^{-\lambda} \tag{1}$$

where λ can be expressed in terms of other known exponents of percolation as [3]

$$\lambda = 1/(\tau - 1). \tag{2}$$

Here $\tau = 1+d/D$ where d is the spatial dimension. It was also argued that the above behaviour is only observed in the asymptotic limit $r \to \infty$. The $\langle M/L^D \rangle$ versus r curve actually changes its slope slowly (in a log–log plot). Hence, for a given range of r, one can define an effective $\lambda_{\mathrm{eff}}(r)$ with $\lambda_{\mathrm{eff}}(r \to \infty)$ given by (2). Very large rank would essentially mean clusters of size one or two in a finite lattice and these are not of present interest.

The distribution function and its moments are useful for studying important properties of a system such as multifractality, lacunarity etc. The distribution of the size of *all* the clusters, which is essentially the number of clusters of a given size (as a function of the size) in a dilute lattice, is well known [1] both at and away from criticality. Distributions of several other quantities such as the size of the spanning clusters, chemical distances, shortest and longest

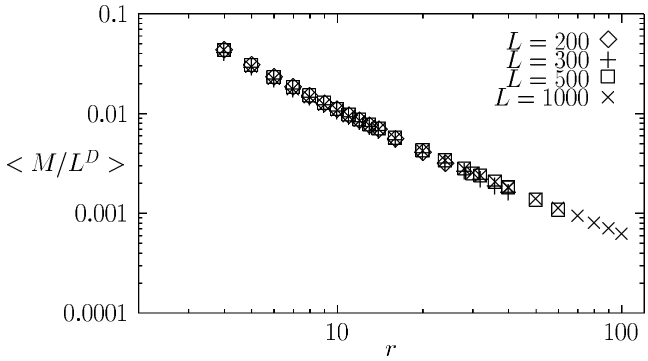

Figure 1. Variation of the average scaled mass $\langle M/L^D \rangle$ against the ranks are shown for lattice sizes $L = 200, 300, 500$ and 1000. A larger number of data (corresponding to higher ranks) are available for increasing lattice size.

paths on the percolation cluster etc, have also been studied in detail [4–9]. In general, at criticality, when the randomness is relevant, a universal non-Gaussian distribution function will exist [10]. Although questions about distribution functions have been addressed for quite some time, a proper understanding is still lacking in several areas [4, 7–9, 11]. Recently, the behaviour of the distribution of the largest cluster below criticality was also studied [12].

The existence of universal scaling functions for the different quantities in the percolating lattice and the properties of the ranked clusters inspired us to study the distribution of the mass or size of these clusters at the percolation threshold. Although most of the quantities which have been studied earlier are directly related to the percolating or spanning cluster (such as the mass of the percolating cluster, the mass of the backbone, the shortest path on the backbone, etc) the smaller clusters are no less important. In addition, the remarkable fractal-like behaviour of the ranked clusters calls for further investigation. Our interest is particularly focused on the question of universality of the distribution function.

In the simulation, the clusters in a square lattice (with helical boundary condition) are identified using the Hoshen–Kopelman algorithm. We rank the clusters at the percolation threshold irrespective of whether the lattice is actually percolating or not. It may be noted that the ranked clusters may have degeneracy in the sense that there may be several clusters with the same rank in a particular realization of the lattice. We checked, however, that incorporating this degeneracy hardly affects the results.

We first check the fact that the slope of the average cluster mass $\langle M/L^D \rangle$ versus r in a log–log plot is indeed not unique in spite of (1) and in agreement with [3]. We also verify that $\lambda_{\rm eff}(r)$ has very weak finite-size dependence, if any, as shown in figure 1. For $r > 30$, there is apparently some size dependence, but for the small lattices (e.g. $L = 200, 300$ etc), such ranks correspond to clusters which are not effectively spanning. Indeed, in [3], the asymptotic value of λ was found from very large lattices. However, one can obtain useful information as long as distribution functions are concerned, even from relatively smaller lattices.

The number of clusters of rank r with mass M/L^D is evaluated. The normalized probability distribution ($P(M/L^D, r)$) of a cluster of scaled mass M/L^D and rank r is obtained by dividing this number by the total number of clusters of rank r. This is shown for the ranks 4, 6, 10 and 14 for several lattice sizes in figure 2. As in [7], where only distributions for the case $r = 1$ were considered, the bin sizes are proportional to $1/L^D$, and one directly obtains a universal distribution for $P(M/L^D, r)$ for several values of L. Another interesting feature is that, as one plots $P(x = M/L^D, r)$ for several ranks, it is found that the peaks of the distribution functions

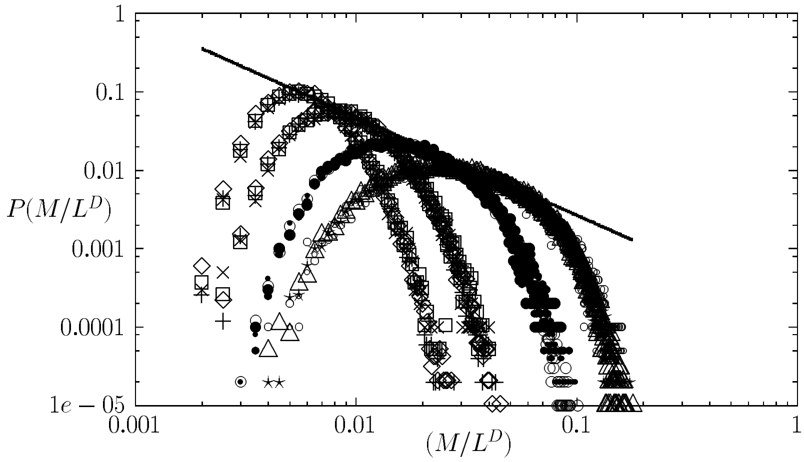

Figure 2. The probability distribution for the ranked cluster masses are shown for ranks $r = 4, 6, 10$ and 14 (from right to left) for lattice sizes $L = 200, 300, 400$ and 500 against the scaled masses. The peaks of the distribution functions show a power law behaviour with M/L^D where the maxima occur for each rank.

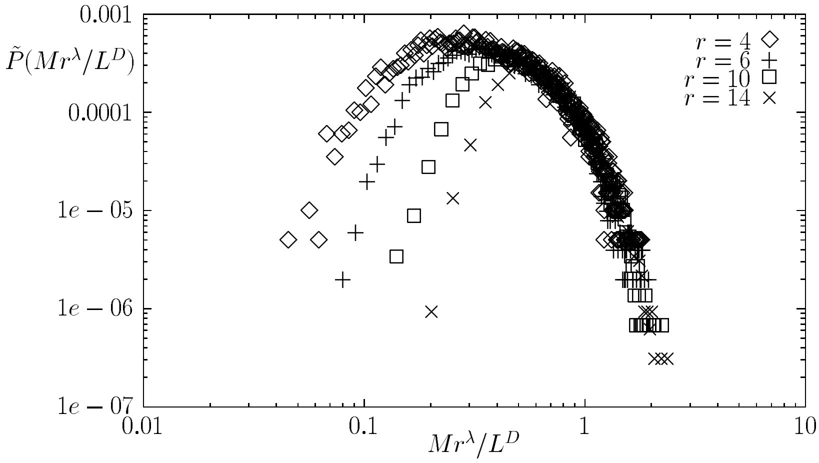

Figure 3. The partial collapse of the data for the scaled distribution $\tilde{P}(Mr^y/L^D) = P(M/L^D, r)r^{-y\zeta}$ is shown for ranks $r = 4, 6, 10$ and 14. The value of y is 1.75.

behave as $P_{\max}(x_{\max}) = x_{\max}^{-\zeta}$ where $x_{\max}$ is the value of x at which the peak occurs. (This is shown by the straight line touching the peaks of the distribution in figure 2 in a log–log plot.) This behaviour of the peaks persists with a rank-independent value of $\zeta \simeq 1.25$ even for the higher ranks.

We are more interested, however, in the behaviour of the probability distribution functions for different ranks for the same lattice size. The peak of the distribution $P(M/L^D, r)$ has a functional dependence on r as $r^{\zeta\lambda_{\text{eff}}(r)}$ from the above-mentioned behaviour and equation (1). However, in general the behaviour of the entire distribution may not be as $r^{\zeta\lambda_{\text{eff}}(r)}$ and we observe that it is better to expect a general form as

$$P(M/L^D, r)r^{-y\zeta} \sim g(Mr^y/L^D) \tag{3}$$

when plotted against the natural scaling argument Mr^y/L^D. Here y is expected to be close

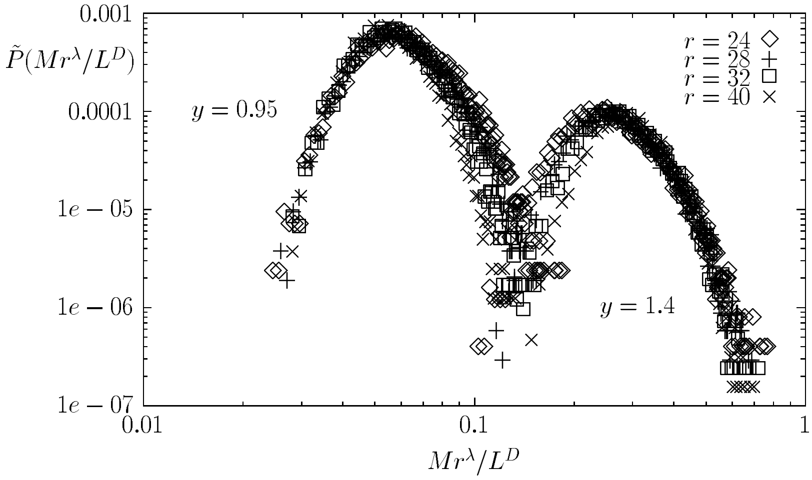

Figure 4. The partial collapses of the data for the scaled distribution $\tilde{P}(Mr^y/L^D) = P(M/L^D)r^{-y\zeta}$ for ranks $r = 24$, 28, 32 and 40 with two different values of y are shown separately.

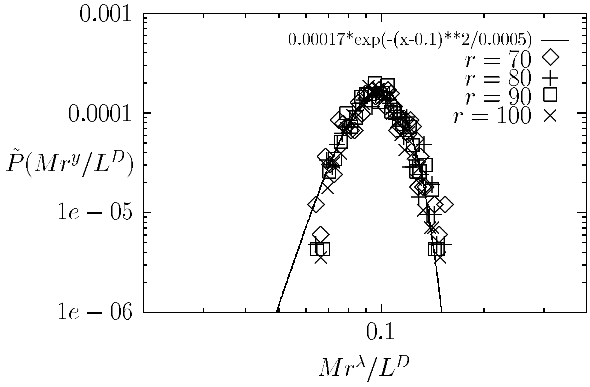

Figure 5. The collapse of the data for the scaled distribution $\tilde{P}(Mr^y/L^D) = P(M/L^D, r)r^{-y\zeta}$ is shown for the ranks $r = 70, 80, 90$ and 100. The value of y is 1.1.

to $\lambda_{\text{eff}}(r)$. The values of y are compared with different values of $\lambda_{\text{eff}}(r)$ (corresponding to three different ranges of r) where the latter are obtained from piecewise least-square fitting of $\langle M/L^D\rangle$ versus r curves. g is a universal scaling function. We attempt to check whether one can actually obtain such a universal function for the distributions.

While the data for the smaller ranges of r are taken from a system of $L = 500$, those in figure 5 correspond to that with $L = 1000$. The number of random configurations generated are 10^4 and 10^3, respectively, for the two sizes. The results for the different ranges of r (as appropriate to the system sizes considered) are summarized below.

Small r. For $4 < r < 14$, we find that only one part of the curves (that beyond the peak value of $P(x, r)$) are collapsing when plotted against the proposed scaling argument with $y \simeq 1.75$. Here the actual value of $\lambda_{\text{eff}}(r)$ is around 1.45.

Intermediate r. For r values in a higher range ($24 < r < 40$), we find that the two parts of the curves collapse separately with different values of y; $y \simeq 1.4$ for the portion beyond the peak, and $y \simeq 0.95$ for the other portion. The value of λ_{eff} for this range of r is found to be close to 1.25.

Higher r. Plotting the scaled probabilities for even higher values of r we find, for the first time, a simultaneous collapse of both sides of the curves with y between 1 and 1.1. The value of λ_{eff} in this range is also $\simeq$1.1. Hence a universal function is indeed obtained for large r values. We believe that, for even higher ranges of r (for which reliable data can be obtained from larger lattices), the same behaviour will persist, with the value of y approaching the asymptotic value of λ. Interestingly, for the smaller and higher ranges of r, y is neither equal to λ_{eff} or the asymptotic value of λ (at least for large x). However, in the scaling regime (i.e. for the large ranks), $y \simeq \lambda_{\mathrm{eff}}$.

The major portion of the universal distribution seems to fit well with a Gaussian distribution function of the form $\exp(-(x-x_0)^2/\sigma)$ with $0 < x < \infty, \sigma \simeq 0.0005$, $A \simeq 0.17$ and $x_0 \simeq 0.1$.

Hence we obtain a distribution function in the following form:

$$P(M/L^D, r) \sim r^{y\zeta} \exp(-(Mr^y/L^D - 0.1)^2)/\sigma) \tag{4}$$

with $y \simeq 1.1$ and $\zeta \simeq 1.25$ for the higher ranks.

Hence we obtain two most significant results in the present study:

(a) The exponent $\zeta \simeq 1.25$ for all ranges of the ranks. This is significant as, while other properties of the system are rank dependent, this particular one remains constant.

(b) The existence of a Gaussian distribution. Most of the distribution functions studied earlier have yielded a more complicated universal function [4, 5, 7]. However, here also the data corresponding to very small values of Mr^y/L^D do not fall on the Gaussian fitting curve.

It is difficult to relate ζ to the known exponents in percolation. Naively, if (1) is to be derived from (4), then

$$\langle M/L^D \rangle = \int \frac{M}{L^D} P(M/L^D, r)\, \mathrm{d}\left(\frac{M}{L^D}\right) \sim r^{-\lambda} \tag{5}$$

gives $\zeta = 1.0$ with $y = \lambda$. This involves the approximation that the mass of the cluster varies from zero to infinity. This approximation and also possible deviations from the Gaussian distribution may be responsible for the discrepancy between this value and the obtained value of ζ; or it may simply be due to errors in numerical estimate.

As already mentioned, the distribution for the probability (per site) of clusters with s sites is known to be $Q(s) \sim s^{-\tau}$ in a percolating lattice. One may expect that this behaviour can be extracted from $P(M/L^D, r)$ by calculating $\Sigma_r P(M/L^D, r)$ as $s = (M/L^D)L^{D-d}$, and a theoretical estimate of ζ can be made. However, it has been numerically verified that one needs to include clusters of all ranks to obtain $Q(s)$ in the above manner, and the absence of a universal scaling law for all r thus does not allow one to theoretically estimate ζ.

In conclusion, the fact that the average cluster size only approached a rank-independent scaling form given by (4) for large r is consistent with our result that the universal form is obtained, again, only in the large-r limit. One needs an exponent y to obtain a collapse of the data which should apparently equal λ. However, y is greater than λ_{eff} for the lower rank ranges. Surprisingly though, for the intermediate rank range, a data collapse is achieved for the smaller masses with a value of y very close to the asymptotic value of λ. It is not clear how significant this equivalence is and whether it is purely accidental. In addition, we get an exponent ζ from the scaling behaviour of the probability distribution which is independent of the rank. An approximate estimate of ζ is attempted for comparison with the numerically obtained value. As in the cases of other quantities in percolation, here a universal function is also seen to exist, which in contrast to the others is a simple Gaussian. The universal functions existing for each rank separately for several system sizes, however, have a more complicated nature.

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